# SOME OPEN PROBLEMS IN RANDOM MATRIX THEORY AND THE THEORY OF INTEGRABLE SYSTEMS

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Abstract. We describe a list of open problems in random matrix theory and integrable systems which was presented at the conference "Integrable Systems, Random Matrices, and Applications" at the Courant Institute in May 2006.

## Introduction

During the conference "Integrable Systems, Random Matrices, and Applications," held at the Courant Institute in May 2006, the organizers asked me to present a list of unsolved problems. What follows is, more or less, the list of problems I presented, written down in an informal style. Detailed references are readily available on the web. In the text, various authors are mentioned by name: this is to aid the reader in researching the problem at hand, and is not meant in any way to be a detailed, historical account of the development of the field. I ask the reader for his'r indulgence on this score. In addition, the list of problems is not meant to be complete or definitive: it is simply an (unordered) collection of problems that I think are important and interesting, and which I would very much like to see solved.

**Problem 1. KdV with almost periodic initial data.** Consider the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

with initial data

$$u(x, t = 0) = u_0(x), \qquad x \in \mathbb{R}\,. \tag{2}$$

In the 1970s, McKean and Trubowitz proved the remarkable result that if the initial data $u_0$ is periodic, $u_0(x+p) = u_0(x)$ for some $p > 0$, then the solution $u(x,t)$ of (1) is almost periodic in time. The conjecture is that the same result is true if $u_0(x)$ is almost periodic. This turns out to be an extremely difficult problem: for such initial data, even short time existence for (1) is not known. Loosely speaking, the difficulty centers around the fact that, in the periodic case, various $L^2$ quantities

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are conserved, e.g.

$$\int_0^p u^2(x,t)\,dx = \int_0^p u_0^2(x)\,dx, \qquad t \geq 0,$$

but in the almost periodic case, we only know that averaged $L^2$ quantities, such as $\lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} u^2(x,t)dx$, are conserved. It is not clear how to use such averaged $L^2$ quantities to analyze the Cauchy problem for (1). Furthermore, the result of McKean and Trubowitz relies in an essential way on the integrability of KdV: in particular, $H(t) = -\frac{d^2}{dx^2} + u(x,t)$ undergoes an isospectral deformation under (1), and the periodic spectrum of $H(t)$ provides integrals for the flow. In the almost periodic case, assuming that a solution $u(x,t)$ exists for $t > 0$, the same is true for $\mathrm{spec}(H(t))$, which equals $\mathrm{spec}(H(t=0))$; the problem is that very little is known about $\mathrm{spec}(H(t=0))$ for general, almost periodic $u_0(x)$. The spectrum may have absolute continuous, singular continuous, and pure point components. For example, in famous work from the 1970s, Dinaburg and Sinai used KAM techniques to show that $H = -\frac{d^2}{dx^2} + \cos x + \cos(\sqrt{2}x)$ has (some) absolutely continuous spectrum. And in the 1980s Moser constructed an example of a limit periodic potential $u_0(x)$ for which $H = -\frac{d^2}{dx^2} + u_0(x)$ has Cantor spectrum. And even if the spectral theory of $-\frac{d^2}{dx^2} + u_0(x)$ for general almost periodic $u_0(x)$ was known, that would only be the first step in understanding the short time, and then the long time, behavior of the solution of (1).

**Problem 2. Universality for Random Matrix Theory (RMT).** Universality for unitary matrix ensembles ($\beta = 2$) in the bulk and at the hard and soft edges is now rather well understood. For orthogonal and symplectic ensembles ($\beta = 1$ and $\beta = 4$ respectively) universality in the bulk and at the hard and soft edges is now understood in the case that the underlying distribution is of the form $\frac{1}{Z_N} e^{-\operatorname{tr} Q(M)}\,dM$, where

$$Q(x) = a_0 x^m + a_1 x^{m-1} + \cdots \tag{3}$$

is a polynomial. It is of considerable interest to prove universality in the cases $\beta = 1$ and $\beta = 4$ for more general potentials $Q(x)$, in particular for $Q(x)$ of the form

$$Q(x) = NV(x), \tag{4}$$

where $V(x)$, is, say, a polynomial. In case (3), the associated equilibrium measure is supported on a single interval, whereas in case (4) the equilibrium measure may be supported on a union of disjoint intervals, and this makes the problem considerably harder. Following Tracy-Widom and Widom, one still utilizes known asymptotics for polynomials orthogonal with respect to the weight $e^{-NV(x)}\,dx$ on $\mathbb{R}$, but now one works on a Riemann surface of genus $g > 0$ as opposed to the Riemann sphere in case (3). The real difficulty involves a certain determinant, $D_m$ say, that arises in the analysis. From the algebraic point of view, $D_m$ can be viewed as controlling the change of basis from orthogonal polynomials to skew orthogonal polynomials, as the size $N$ of the polynomials goes to infinity. From an analytical/physical point of view, $D_m$ arises in the computation of a ratio of partition functions, again in the large $N$ limit. The essential task here is to show that $D_m \neq 0$. In case (3) this follows from a lengthy ad hoc calculation; in the case (4), the problem is wide open, though recent work of Shcherbina may point the way. Alternatively, Lubinsky

has recently developed very interesting new methods for unitary ensembles. Can these methods be adapted to prove universality for the orthogonal and symplectic ensembles?

For Wigner ensembles, universality at the edge (Soshnikov,...) is now well understood for a wide variety of distributions on the entries of the matrices. It is a long-standing conjecture, now more than forty years old, that universality is also true in the bulk, but there has been very little progress. At the edge, moment methods are powerful enough to prove universality, but in the bulk they provide no information. By contrast, for invariant ensembles of random matrices, one is able to prove universality in the bulk because of the availability of explicit formulae for eigenvalue statistics that are amenable to asymptotic analysis.

A priori, universality is more plausible for unitary, orthogonal and symplectic ensembles because of the invariance properties that are built in: such ensembles are already "part of the way there." This is not the case, however, for Wigner ensembles. Universality in the bulk for Wigner matrices is a conjecture par excellence that digs deep into the structure of random matrices. Numerical experiments provide convincing evidence that it is true.

**Problem 3. Riemann-Hilbert Problem with non-analytic data.** In many situations one is concerned with the asymptotic behavior of Riemann-Hilbert problems with exponentially varying data of the form $e^{in\phi(z)}r(z)$, $n \to \infty$. The Deift-Zhou nonlinear steepest descent method for such problems requires $\phi(z)$ to be analytic. The analyticity is used in two ways: to control the equilibrium measure associated with the problem, and then to deform the contour for the RHP. By contrast, the method only requires minimal smoothness for $r(z)$ (see eg. Deift-Zhou in the context of Problem 12 below). It is of considerable theoretical and practical interest to extend the nonlinear steepest descent method to situations where $\phi$ is no longer analytic, and has, for example, only a finite number of derivatives. For very interesting work on the analyticity problem, we refer the reader to a recent paper of Miller and McLaughlin. There is also interesting, older work due to Varzugin.

**Problem 4. Painlevé equations.** What I have in mind here is not a specific problem, but a project, a very large scale project. The six (nonlinear) Painlevé equations form the core of "modern special function theory." The role that the classical special functions, such as the Airy, Bessel, and Legendre functions, started to play in the $19^{\text{th}}$ century, has now been greatly expanded by the Painlevé functions. Increasingly, as nonlinear science develops, people are finding that the solutions to an extraordinarily broad array of scientific problems, from neutron scattering theory, to PDEs, to transportation problems, to combinatorics,..., can be expressed in terms of Painlevé functions. What is needed is a project, similar to the Bateman project, or a new volume of Abramowitz and Stegun, devoted to the Painlevé equations. Much can be, and has been, proved regarding the algebraic and asymptotic properties of Painlevé functions. Here the role of integral representations and the classical steepest descent method in deriving precise asymptotics and connection formulae for the classical special functions is played, and expanded, by a Riemann-Hilbert representation of the Painlevé equations, together with the noncommutative steepest descent method introduced in 1993. Very little is known, however, beyond ad hoc calculations, about the numerical solution of the Painlevé

equations. If $u(x)$ is the solution of the Painlevé II equation, say, which is asymptotic to the Airy function $Ai(x)$ as $x \to +\infty$, one would like to know, for example, the location of its poles in the complex $x$-plane. A modern "Bateman Project: Painlevé equations" would not/should not provide tables for such solutions. Rather, it should provide reliable, easy to use software to compute the solutions. Writing useful software for such nonlinear equations presents many challenges, conceptual, philosophical and technical. Without the help of linearity, it is not at all clear how to select a broad enough class of "representative problems." The software should be in the form of a living document where new numerical problems can be addressed by a pool of experts as they arise. And at the technical level, how does one combine asymptotic information about the solutions obtained from the Riemann-Hilbert problem, together with efficient numerical codes in order to compute the solution $u(x)$ at finite values of $x$?

I believe that the importance of the Painlevé Project will only grow with time. It should be viewed as creating a national resource and should probably be funded and led at the national level. The NIST Project "Digital Library of Mathematical Functions", where Peter Clarkson has a contribution on Painlevé functions, is an encouraging first step.

**Problem 5. Multivariate analysis.** Random matrix theory was introduced into theoretical physics by Wigner in the 1950s in his study of neutron scattering resonances, but as a subject, RMT goes back to the work of statisticians at the beginning of the 20th century. Recently, advances in RMT have opened the way to the statistical analysis of data sets in cases where the number of variables is comparable to the number of samples, and both are large. One might, for example, be interested in the daily temperature in hundreds of cities around the world, over a 365 day time period. At the technical level, one considers the statistics of the singular values of (appropriately centered and scaled) $p \times n$ matrices $M = (M_{ij})$, where $p \sim n \to \infty$. Here $p$ is the number of variables and $n$ is the sample size. More precisely, one centers the $M_{ij}$'s around their sample averages,

$$M_{ij} \to \hat{M}_{ij} = M_{ij} - \frac{1}{n}\sum_{k=1}^{n} M_{ik},$$

and considers the eigenvalues $l_1 \geq \cdots \geq l_p \geq 0$ and associated eigenvectors $w_1, \ldots, w_p$ of the $p \times p$ sample matrix $S = \frac{1}{n}\hat{M}\hat{M}^T$. The $l_i$'s and $w_i$'s are known as the principal component eigenvalues and eigenvectors, respectively. In Principal Component Analysis (PCA) "significant" dimension reduction in the data occurs if the first few principal components $l_1, l_2, \ldots$ account for a "high" proportion of the total variance $\operatorname{tr} S = \sum_{j=1}^{p} l_j$.

A common model for the variables $M_{ij}$ is to assume that they follow a (real) $p$-variate Gaussian distribution $N_p(\mu, \Sigma)$ with mean $\mu$ and covariance matrix $\Sigma$. Thus the columns $(M_{1j}, M_{2j}, \ldots, M_{pj})^T$ provide $n$ independent samples for $N_p(\mu, \Sigma)$. Using recent results from RMT, much has now been proved about the statistics of $l_1, l_2, \ldots$ as $p, n \to \infty$, $p/n \to \gamma \in (0, \infty)$, in the case $\Sigma = I$. In particular, we know that in the limit, $l_1$, appropriately centered and scaled, satisfies the Tracy-Widom distribution for the largest eigenvalues of a GOE matrix. However, most interesting applications involve so-called spiked populations, a terminology introduced by Johnstone, i.e. situations where most of the eigenvalues $\eta_1, \ldots, \eta_p$ of $\Sigma$ are equal to

a common value, say 1, but the first few eigenvalues are greater than 1. Thus

$$\eta_1 \geq \eta_2 \geq \cdots \geq \eta_k > \eta_{k+1} = \cdots = \eta_p = 1$$

for some fixed $k << p$. It is a major problem in multivariate analysis to analyze the statistics of the eigenvalues $l_1, l_2, ...$ as $p, n \to \infty$, $p/n \to \gamma \in (0, \infty)$ for such spiked populations. There are very interesting phase transitions in the theory. For example if $\eta_1 > 1+\sqrt{\gamma}$, then, as $p \sim n \to \infty$, $l_1$ emerges from the Marchenko-Pastur continuum $\big((\sqrt{\gamma}-1)^2, (\sqrt{\gamma}+1)^2\big)$, where most of the $l_j$'s tend to accumulate, and almost surely

$$l_1 \to \eta_1 \cdot \left(1 + \frac{\gamma}{\eta_1 - 1}\right) > (1+\sqrt{\gamma})^2 \,.$$

In the spiked, complex case, i.e. when the columns $(M_{1j}, M_{2j}, ..., M_{pj})^T$ are sampled from the complex $p$-variate Gaussian distribution, much is known about the asymptotic distribution of the $l_j$'s, as $p, n \to \infty$, $p/n \to \gamma \in (0, \infty)$. By contrast in the real case, apart from a.s. convergence of the $l_i$'s, very little is known about their asymptotic distributions. In the spiked, complex case the analysis is enabled by a particular technical tool, the Harish-Chandra-Itzykson-Zuber formula: unfortunately, no analog of this formula is known in the real case. While there are relatively few applications of complex spiked populations, knowledge of the asymptotic distributions of the $l_i$'s for real spiked populations would have immediate applications to a wide variety of problems in signal processing, genetics and finance.

**Problem 6.** **$\beta$-ensembles.** Random point processes corresponding to $\beta$-ensembles, or, equivalently, log gases at inverse temperature $\beta$, are defined for arbitrary $\beta > 0$. The orthogonal, unitary, and symplectic ensembles corresponding to $\beta = 1, 2$, or 4, respectively, are now, of course, well understood, but other values of $\beta$ are also believed to be relevant in applications, for example, in the statistical description of headway in freeway traffic. For certain rational values of $\beta$, $\beta$-ensembles are related to Jack polynomials, but for general $\beta$ much less is known. The analysis of $\beta$-ensembles for general $\beta$ represents an interesting, and increasingly important, challenge.

Recently there have been significant developments in the theory of general $\beta$-ensembles. As a result of the work of Edelman and Dumitriu, and also others, we now know that for all $\beta > 0$, there exist (tridiagonal) random matrix models whose eigenvalues are distributed according to $\beta$-ensembles. Furthermore, taking an appropriate scaling limit of these tridiagonal matrix ensembles, one arrives at the following remarkable fact. Let $B(x)$, $x \geq 0$, denote Brownian motion, and for any $\beta > 0$, let $H_\beta$ denote the Schrödinger operator $\frac{d^2}{dx^2} - x - \frac{2}{\sqrt{\beta}} dB(x)$ acting on $L^2((0,\infty), dx)$ with Dirichlet boundary conditions at $x = 0$. Then (Edelman-Sutton, Ramirez-Rider-Virag) for almost all realizations $B(x)$, $x \geq 0$, $H_\beta$ is self-adjoint with discrete spectrum $\lambda_1(B,\beta) > \lambda_2(B,\beta) > \cdots$ and for each $k$, $\lambda_k(B,\beta)$ has precisely the same distribution as the $k^{\text{th}}$ largest eigenvalues of the corresponding $\beta$-ensemble in the standard edge scaling limit. Part of the challenge in analyzing general $\beta$-ensembles is to use $H_\beta$ to obtain information about these ensembles. Already the variational characterization of the $\lambda_k$'s has been used to give simple proofs of bounds on the asymptotics of the distributions of the $\lambda_k$'s. Question: can one derive the Tracy-Widom formula for $\lambda_1(B; \beta = 2)$, say, directly from $H_\beta$?

At a more conceptual level, the ($\beta$-ensemble $\leftrightarrow H_\beta$) correspondence brings random matrix theory front and center into the arena and practice of modern day probability theory.

**Problem 7. Non-self adjoint spectral problems.** Much is now known, theoretically and also numerically, about the spectrum of self-adjoint operators. This is in great contrast to the situation regarding non-self-adjoint operators, where the spectral theory is far more subtle and numerical schemes must overcome significant, inherent instabilities. The heart of the difficulties and instabilities lies in the following fact: in the self-adjoint case $A = A^*$, the resolvent $(A-\lambda)^{-1}$ is bounded by the distance of $\lambda$ to the spectrum of $A$, but in the non-self-adjoint case, this is no longer true, as we see already in the $2 \times 2$ case, $A = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$, $n \to \infty$. In recent years, a number of authors (e.g. Trefethan, Davies, ...) have initiated a systematic approach to non-self-adjoint spectral problems, notions such as pseudospectrum have come into prominence, and other authors have conducted in depth studies of particular non-self-adjoint spectrum problems which arise in practice. For example, in analyzing the semi-classical limit of the focusing NLS equation, one must analyze the spectrum of the associated AKNS operator $T(h)$, as Planck's constant $h$ goes to zero. The operator is non-self-adjoint, and as $h$ goes to zero more and more eigenvalues, corresponding to solitons, emerge in the complex plane. It is of critical importance to the analysis of the semi-classical limits for NLS to determine where in the plane, and at what rate, the eigenvalues accumulate. The difficulty in doing this, theoretically and numerically, is illustrated by the following fact: the spectrum of $T(h)$ off the real axis is a discrete set, whereas the numerical range of $T(h)$ is an open subset of the plane. Nevertheless, every point $\lambda$ lying in the numerical range of $T(h)$ is an eigenvalue of $T(h)$ to all orders in $h$, $\|(T(h)-\lambda)u\| = O(|h|^k)$ for any $k \geq 1$, for some $u = u(h)$, $\|u\| = 1$. In the language of Kruskal, the computation of the spectrum of $T(h)$ is a problem "beyond all orders." Much has been done (Kamvissis-Miller-McLaughlin, Tovbis-Venakides-Zhou,...) in analyzing the semi-classical limit for NLS in special situations where the spectral problem can be solved explicitly. The spectral problem with general data, both for NLS and also other related non-self-adjoint problems, however, is far from understood, and poses a great challenge whose resolution is still only in the initial stages.

**Problem 8. Long-time behavior with non-generic initial data.** The long-time behavior of the solution of the Cauchy problem for a great many integrable systems on the line is now well-understood, using, for example, the Riemann-Hilbert/steepest descent method. The method depends on full knowledge of the nature of the spectrum of the associated Lax operator. For systems such as KdV, MKdV, defocusing NLS and the Toda lattice, for example, one is able to describe the solution asymptotically in complete detail for general initial data. But for focusing NLS, where the associated AKNS Lax operator $T$ is non-self-adjoint (here we make contact with Problem 7), the situation is different. For generic $T$ (i.e. an open dense set of $T$'s in any reasonable topology) the spectrum consists of the real line, where the spectrum is absolutely continuous, together with a finite number of simple eigenvalues (corresponding to solitons) off the real axis. The analysis of the long-time behavior of focusing NLS with such generic initial data proceeds in a straightforward manner similar to KdV, MKdV, etc. For general initial data,

however, the situation is more complicated. For example, let $z_0 > 0$ be any positive number and let $D$ be any arbitrarily small open disk in the complex plane centered at $z_0$, such that $D \subset \{z : \Re z > 0\}$. Let $D^+$ denote the intersection of $D$ with the upper half plane, and let $u(z)$ be an arbitrary function analytic in $D^+$, and continuous in $\overline{D^+}$. Let $B = \{z \in D^+ : u(z) = 0\}$. Then there exists (Zhou) an AKNS operator $T$ with infinitely smooth, rapidly decaying coefficients with the property that each point in $B$ is an $L^2(\mathbb{R})$-eigenvalue of $T$. In other words, there exist (non-generic) operators $T$ with Schwartz space coefficients which have $L^2$ spectrum accumulating on the real line at an essentially arbitrary rate. It is a very interesting question to determine what effect such singularities would have on the long-time behavior of the solution of NLS. In particular, recalling that focusing NLS provides a model for data transmission along communication cables, are the effects measurable?

The difficulty that we encounter here is not limited to situations where the associated Lax operator is non-self-adjoint. Even in situations when the associated operator is self-adjoint, but of order greater than two, similar difficulties can arise. This is true, in particular, for the Boussinesq equation, where the associated Lax operator is third order. The long-time behavior of the solutions of the Boussinesq equation with general initial data is a very interesting problem with many challenges. Even in the case with generic initial data the situation is only partially understood.

**Problem 9. The parking problem.** A number of so-called "transportation" problems have now been analyzed in terms of RMT. These include: the "vicious" walker problem of M. Fisher, the bus problem in Cuernavaca, Mexico, the headway traffic problem on highways, and the airline boarding problem of Bachmat et al. Recently, researchers in London, Prague, and also Ann Arbor, have noticed an intriguing phenomenon. They found that the fluctuations in the spacings between cars parked on a long street exhibited RMT behavior. Furthermore, Šeba found that there was a difference whether the street is two-way or one-way (On a two-way street, the cars park only on the right, while on a one-way street one of course has the option of also parking on the left.) Quite remarkably, for two-way streets Šeba found GUE statistics, but for left-side parking on one-way streets he found GOE statistics. It is a great challenge to develop a microscopic model for the parking problem, in analogy, perhaps, with the microscopic model introduced by Baik et al. to explain the RMT statistics for the bus problem in Cuernavaca. Šeba's recent, intriguing calculations on the parking problem can be found posted on the web.

**Problem 10. A Tracy-Widom Central Limit Theorem.** The fact that RMT, and the Tracy-Widom distributions, arise in so many problems in so many different areas leads one to the following question: how can one characterize RMT in purely probabilistic terms? For example, we know that if we take i.i.d.'s $(a_1, a_2, ...)$, add them up, and then center and scale appropriately,

$$(a_1, a_2, ...) \to (S_1, S_2, ...), \qquad S_n = \frac{\sum_{i=1}^n a_i - n\mu}{\sqrt{n}},$$

then as $n \to \infty$, $S_n$ converges in distribution to a Gaussian random variable: this is the famous Central Limit Theorem. The analogous situation for RMT is the

following: take i.i.d.'s $(a_1, a_2, ...)$, perform an operation $X$ on them,

$$(a_1, a_2, ...) \to (X_1, X_2, ...),$$

and as $n \to \infty$ the $X_n$'s converge to the Gaudin distribution, or the Tracy-Widom distribution. The question is, "What is $X$?" Important progress towards answering this question has been made recently, and independently, by Baik-Suidan and Bodineau-Martin, but the full problem remains open and very challenging.

**Problem 11. The Toda lattice with random initial data.** A fundamental question in numerical analysis is the following: how long does it take on average to diagonalize a random symmetric $n \times n$ matrix $M$? There are different opinions about what is meant by random, and also many eigenvalue algorithms that one could choose. Also one must specify the accuracy that is required. For definitiveness, let us assume that $M$ is chosen from GOE and that we use the standard QR eigenvalue algorithm. The fundamental question then takes the following more concrete form: given $\epsilon > 0$, how many QR steps does it take on average to compute the eigenvalues of a GOE random matrix to order $\epsilon$? In practice, one never computes the eigenvalues of a full $n \times n$ matrix $M$. Rather one first reduces $M$ to a tridiagonal matrix

$$J = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix}$$

with the same spectrum as $M$ using, say, a succession of Householder transformations. Now it turns out, by pure serendipity, that the Householder transformation is eminently compatible with GOE and the statistics of $J$ can be computed explicitly: the $a_i$'s and $b_j$'s are independent, the $a_i$'s are i.i.d. Gaussians, and the $b_j$'s have $\chi$ distributions, $\chi_{n-j}$, $1 \le j \le n-1$. Such matrices are an example of what we may call a TE1 (triangular ensemble, $\beta = 1$). With these comments, the fundamental question now takes the following sharp form: given $\epsilon > 0$, how many QR steps does it take on average to compute the eigenvalues of a TE1 matrix $J$ to order $\epsilon$?

In another, independent development in the 1970s, Flaschka, and also Manakov, showed that the Toda lattice, consisting of $n$ 1-dimensional particles interacting with exponential forces

$$\ddot{x}_k = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}}, \qquad 1 \le k \le n, \tag{5}$$

$(x_0 \equiv -\infty, x_{n+1} \equiv +\infty)$ is a completely integrable Hamiltonian system with a Lax operator given by the tri-diagonal matrix

$$J = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix},$$

where

$$\begin{cases} a_i = -\frac{1}{2}\dot{x}_i, & 1 \le i \le n \\ b_i = \frac{1}{2}e^{\frac{1}{2}(x_i - x_{i+1})}, & 1 \le i \le n-1. \end{cases}$$

With these variables, (5) takes the isospectral form

$$\frac{dJ}{dt} = [B, J] = BJ - JB, \tag{6}$$

where

$$B = \begin{bmatrix} 0 & b_1 & & \\ -b_1 & 0 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & -b_{n-1} & 0 \end{bmatrix}.$$

Subsequently Moser integrated (5) explicitly and showed that as $t \to \infty$ the particles become free,

$$\begin{cases} \dot{x}_i(t) = \dot{x}_i(\infty) + o(1), & 1 \le i \le n \\ x_i(t) = t\dot{x}_i(\infty) + x_i(\infty) + o(1), & 1 \le i \le n, \end{cases}$$

for some constants $\{x_i(\infty)\}_{i=1}^n$ and $\{\dot{x}_i(\infty)\}_{i=1}^n$, where $\dot{x}_n(\infty) > \dot{x}_{n-1}(\infty) > \cdots > \dot{x}_1(\infty)$. Said differently, as $t \to \infty$, $J(t)$ converges to a diagonal matrix $\operatorname{diag}\left(-\frac{\dot{x}_1(\infty)}{2}, \ldots, -\frac{\dot{x}_n(\infty)}{2}\right)$. But as $J(t)$ undergoes an iso-spectral evolution, it follows that $a_i(\infty) = -\frac{\dot{x}_i(\infty)}{2}$, $1 \le i \le n$, must be the eigenvalues of the original matrix $J(t = 0)$, where $a_i(0) \equiv -\frac{\dot{x}_i(0)}{2}$, $1 \le i \le n$, $b_j(0) \equiv \frac{1}{2} e^{\frac{1}{2}(x_j(0) - x_{j+1}(0))}$, $1 \le j \le n-1$. Deift-Li-Nanda-Tomei then raised the possibility of using the Toda lattice as an eigenvalue algorithm (the "Toda algorithm"): given a tridiagonal matrix $J_0$ and $\epsilon > 0$, solve (6) with initial condition $J(t = 0) = J_0$, until $\max_{1 \le i \le n} b_i(t) < \epsilon$. Then $a_i(t), 1 \le i \le n$, are the eigenvalues of $J_0$ to order $\epsilon$. Using earlier work of Symes, Deift et al showed that the QR algorithm itself was the time-$k$ map, $k = 1, 2, ...,$ of a completely integrable Hamiltonian system which Poisson commutes with the Toda Hamiltonian. In this picture, the choice of an eigenvalue algorithm becomes simply the choice of a Hamiltonian vector field. In view of the above comments and remarks, the fundamental question becomes: how long does it take on average for Toda particles with TE1 initial data to become free (i.e. $\max_{1 \le i \le n-1} b_i < \epsilon$)? Taking into account the ideas of deflation, it is enough to consider the time $t$ at which just $b_{n-1}(t) < \epsilon$.

The solution of the fundamental question in the above form is a fascinating challenge and would clearly be of great interest to scientists and engineers alike.

**Problem 12. Perturbation theory for infinite dimensional integrable systems.** The bijective mapping properties of the scattering transform for a variety of integrable systems on the line between appropriate weighted Sobolev spaces have now been established (X. Zhou) using Riemann-Hilbert techniques. This has made it possible, in particular, to analyze (Deift-Zhou) the long-time behavior of the solution of the Cauchy problem for a variety of integrable systems in a fixed space without a "loss of derivatives". This in turn has made it possible to analyze perturbations of integrable systems. For example, Deift-Zhou analyzed the perturbed defocusing NLS equation

$$iu_t + u_{xx} - 2|u|^2 u - \varepsilon V(|u|)u = 0 \tag{7}$$

$$u(x, t = 0) = u_0(x) \in H^{1,1} = \{f \in L^2 \,|\, f', xf \in L^2(\mathbb{R})\},$$

where $V(|u|) \sim |u|^p$ as $|u| \to 0$, for some $p > 2$ sufficiently large. In addition to a full description of the long-time behavior of the solution of (7), a rather surprising

outcome of their calculations is that (7) remains completely integrable for all $\varepsilon$ (small and) positive. In the perturbation theory of the linear Schrödinger equation – when the term $-2|u|^2u$ is absent from (7) – the key role in the analysis is played by the Fourier transform which diagonalizes the linear part of the equation. For the full equation (7), the role of the Fourier transform is now played by the scattering transform which "diagonalizes" the (cubic) NLS part of the equation. In the case of the linear Schrödinger equation, the method relies on certain precise estimates that one obtains from the Fourier transform and the classical steepest descent method; in the fully non-linear case (7), the analogs of these estimates are obtained from the Riemann-Hilbert version of the scattering/inverse scattering transform, together with the steepest descent method for RH problems. One shows, in particular, that $\lim_{t\to\infty}(\sup_{x\in\mathbf{R}}|u(x,t)|) = 0$, which implies that as $t \to \infty$ the perturbation term $\varepsilon V(|u|)u \sim \varepsilon|u|^p u$, $p > 2$, becomes small with respect to the NLS term $2|u|^2u$.

It is of great interest to consider perturbations of type (7) for the focusing (cubic) NLS equation, i.e. where the term $-2|u|^2u$ is replaced by $+2|u|^2u$. This changes the problem fundamentally as the focusing equation has soliton solutions. Such solutions do not decay uniformly in time (i.e. $\sup_{t,x\in\mathbb{R}}|u(x,t)| > 0$) and this complicates the analysis of the perturbed equation enormously, as the perturbation term is no longer small with respect to $2|u|^2u$ (see above). Equation (7) in the focusing case is just one example of many and varied systems in PDE/Mathematical Physics. In higher dimensions, the behavior of such perturbed systems in the neighborhood of a simple non-linear bound state is well understood (Soffer, Weinstein,...), but for more than one bound state, very little is known.

The solution of (7) in the focusing case in the neighborhood of a $k$-soliton for (cubic) NLS ($k \geq 2$), together with a detailed description of the long-time asymptotics, would be regarded as a very significant development in the theory of PDEs/Mathematical Physics. In a certain sense, one expects (7) once again to be integrable for small $\varepsilon$. Equation (7) is of course only one of many similar systems one could consider. For example, one could consider whether the Toda shock and Toda rarefaction phenomena persist when the exponential forces of interaction are perturbed: all numerical evidence indicates that this is so. Also, one could consider perturbations of the sine-Gordon equation as a model for the Fermi-Pasta-Ulam problem.

There is a considerable body of work (Kuksin, Kapeller-Pöschel) on Hamiltonian perturbations of integrable systems such as NLS or KdV in the spatially periodic case. Here KAM methods apply and the authors show that certain finite dimensional tori corresponding to finite gap solutions in the integrable case survive under perturbation. The periodic problem is more complicated than the problem on the line because of the action of dispersion. On the whole line dispersion forces soliton-free solutions $u(x,t)$ to decay, $\sup_{x\in\mathbb{R}} u(x,t) \to 0$ as $t \to \infty$, and even for solutions with solitons (cf. comments above) we still expect decay away from the location of the solitons. In the periodic case, dispersion has no "room" to act and the perturbation term $\varepsilon V(|u|)u$ is of the same order as $|u|^2u$ for all time.

**Problem 13. Perturbation theory for exactly solvable combinatorial problems.** The asymptotic behavior of a variety of combinatorial problems has now been analyzed in great detail. Here we have in mind Ulam's problem for the length of the longest increasing subsequence, the tiling problem for the Aztec diamond, the hexagon tiling problem, and the last passage percolation problem,

amongst many others. In all cases, in an appropriate scaling limit, the statistical fluctuations in the systems at hand are described by random matrix theory. The asymptotic analysis, however, depends in a critical, and rigid way, on the underlying probability measures for the systems. For example, the analysis of the last passage percolation problem requires the waiting time $w_i$ at each site $i$ to be either geometrically or exponentially distributed (Johansson): if the statistics of the waiting times is neither geometric nor exponential, the analysis fails completely. What happens if the geometric distribution, say, is slightly perturbed? The challenge here is to develop an effective perturbation theory for such systems. One expects that the random matrix behavior of the fluctuations should persist. Related work on this problem has been done by Baik-Suidan and Bouchard-Martin (cf. Problem 10).

**Problem 14. Initial/boundary value problems for integrable systems.** In the late 1990s Fokas introduced a new, more flexible approach to inverse scattering theory, and in recent years a number of researchers (Fokas, Its, ..., Anne Boutet de Monvel, Shepelsky,...) have applied Fokas' approach to the initial/boundary value problem for various integrable systems such as NLS. Much progress has been made, but there is a basic, and puzzling, obstacle to applying the method, viz. one needs to know certain dependent data in order to proceed. For example, the initial/boundary value problem for NLS is well-posed if one gives the initial data $u(x, 0) = u_0(x)$, $x \geq 0$, and the boundary data $u(0, t) = u_1(t)$, $t \geq 0$. However, in implementing the method it turns out that one needs to know the *dependent* data $u_x(0, t)$, $t \geq 0$, as such information appears explicitly in the solution formulae. Progress has been made in determining $u_x(0, t)$, $t \geq 0$, from $u_0(x)$ and $u_1(t)$ intrinsically via the method, but the control one obtains on $u_x(0, t)$ is not sufficient in order to obtain the long-time behavior of the initial/boundary value problem. This is true even in simple cases, such as the following: suppose $u(x, t)$ solves NLS in $(x \geq 0, t \geq 0)$ with $u(0, t) = \sin(\omega t)$, $\omega \neq 0$, and $u(x, 0) = u_0(x)$, where $u_0$ is smooth with compact support. How does $u(x, t)$ behave as $t \to \infty$? The solution of this problem for any such $u_0(x)$ would be a very significant development in the theory, and would also be of considerable interest in science and engineering.

There is a philosophical point at stake here. The evolution of NLS in $x > 0$ represents the interplay of forces which are "integrable" at some fundamental, algebraic level, and indeed, if no other forces are present, as in the full-line scattering or periodic cases, the equation can be integrated explicitly. When one considers the initial/boundary value problem, however, new physical forces come into play which describe the interaction of the particles on the boundary with the NLS particles in the interior. There is absolutely no a priori guarantee that the enlarged system, "NLS particles in $x > 0$"+"particles on the boundary", is integrable. It may be that the long-time behavior of the composite system can only be solved for "generic, Cantor-set" like data, as is familiar from KAM theory. In other words, an explicit description of the long-time behavior of the solution of the initial/boundary value problem for NLS for general initial data may not be possible. This is a very intriguing situation.

**Problem 15. Multi-matrix models and models with an external field.** There has been considerable progress (Kuijlaars,...) in understanding basic statistics such as the correlation functions for the 2-matrix random matrix model, and

also matrix models with a source. The key element in these developments has been the successful extension by Kuijlaars et al of the Riemann-Hilbert/steepest descent method to $3 \times 3$ Riemann-Hilbert problems. So far only the simplest situations have been considered. In order to consider the generic situation, one must, in particular, extend the Riemann-Hilbert/steepest descent method to $n \times n$ Riemann-Hilbert problems. This is a challenging problem which would have important implications, not only for random matrix models, but also for problems in other areas, such as Padé-Hermite approximations and irrationality questions for distinguished real numbers, and multi-orthogonal polynomials.

**Problem 16. Poisson/Gaudin-Mehta transition.** On the appropriate scale, the bulk eigenvalues of a random GOE matrix $M$ exhibit Gaudin-Mehta statistics. As noted in Problem 2, the same is believed to be true for general Wigner matrices. On the other hand, if $M = M^T$ has i.i.d. entries and bandwidth $W = 1$ (i.e. $M$ is tridiagonal), then, on the appropriate scale, the bulk eigenvalues of $M$ exhibit Poisson statistics. As the bandwidth $W$ increases from 1 to $N - 1 = \dim(M) - 1$, the eigenvalue statistics must change from Poisson to Gaudin-Mehta. A back of the envelope calculation suggests that there should be a (sharp) transition in a narrow region around $W \approx \sqrt{N}$. This is a well-known, outstanding, open conjecture with many implications for theoretical physics, particularly wave propagation in random media. There is a substantial body of work on this problem, which can be easily traced on the web.

There are many other areas, closely related to the problems in the above list, where much progress has been made in recent years, and where much remains to be done. These include: totally asymmetric simple exclusion processes (TASEP) following the work of Borodin, Sasamoto, Ferrari,..., asymmetric simple exclusion processes (ASEP) following the work of Tracy, Widom,..., orthogonal rational functions (Vartanian, Zhou, McLaughlin), partition functions for random matrix models and their connections to mappings on Riemann surfaces (Itzikson, Zuber,..., McLaughlin, Ercolani,..., Guionnet), the representation theory of "large" groups such as the infinite symmetric group or the infinite unitary group (Olshanski, Borodin, Okounkov,...), and many others. But as I said in the beginning, my list is not complete or definitive.

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